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AN ANALYSIS OF THE TRANS-SPECTRAL-COHERENCE FOR DUFFING OSCILLATORS UNDERGOING CHAOS

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A theoretical explanation of the observed Trans-Spectral-Coherence for the signal generated from the Duffing oscillators during chaos has been presented. As expected based on findings from earlier work by M. D. Miller strong coherence was found when the Duffing oscillators gave rise to periodic solutions. When the solution was chaotic, it was expected that there would not be phase coherence at all. This was true for the phase coherence between the fundamental frequency ω and sub-harmonics and the fundamental frequency and even harmonics. However, for the odd harmonics, strong phase coherence between ω and 3ω, 5ω, etc. has been found. In order to explain this phenomenon, closed orbits in chaos have been theoretically analysed. These orbits have been subdivided into many groups. Within each group there is π phase shift which affects the coherence between ω and 2ω frequencies and does not affect the coherence between ω and 3ω frequencies. This closed orbit analysis has been used to explain the numerical results mentioned above. Similar analysis has been carried out to explain why the phase coherence between ω and the sub-harmonics vanishes during chaos.

1. INTRODUCTION

Since the earliest days of the study of chaotic dynamics, the Duffing oscillator has attracted the attention of many researchers [1–2]. In this paper, the phase coherence across the frequency spectrum in the Duffing oscillator has been investigated. Phase information is essential for the understanding of chaos. The purpose of this paper is to study, both experimentally and theoretically, the phase coherence in the Duffing oscillator, especially when the solution is chaotic.

Miller [3] has studied the phase coherence in chaos through the bispectral method. He found that in the sine-Gordon chain, which is closely related to the Duffing oscillator, there existed strong bicoherence when the system was periodic and the bicoherence vanished when the system became chaotic.

Recently, Vaidya and Anderson [4] have presented a signal processing technique: the Trans-Spectral-Coherence technique (TSC). As in the case of bispectra, it explicitly takes advantage of a strong possibility that many of the signals of interest are of non-linear

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origin. Thus, it is a powerful tool for signal processing in non-linear systems. The significant advantage of TSC is that it can easily analyse the interactions of two or more frequencies when their ratio is a rational number. The calculation of the interactions of three or more frequencies is not significantly more difficult than that of two frequencies.

In this paper, the TSC has been applied to analyse the phase coherence in the Duffing oscillator. As expected from earlier work [3], the phase coherence was found when the Duffing oscillator gives rise to periodic solutions. Also as expected from earlier work [3], the phase coherence could not be found between the fundamental frequency ω and the harmonics 2ω, 4ω, etc. when the solution is chaotic. However, the strong phase coherence has been found between ω and 3ω, 5ω, etc. even when the solution is chaotic, as presented at the 121*st Meeting of the Acoustical Society of America* in 1991 [5]. Although the numerical result was reported quite some time ago, from a theoretical point of view the results have puzzled the authors for many years. Specifically, how can such strong coherence persist in chaos and if it does, why does it vanish for the even harmonics. This paper is mainly a theoretical solution of this puzzle. However, the authors believe that it sheds important light on the nature of chaos itself.

In connection with understanding the nature of chaos, the authors believe it is very important to recognize that closed orbits of chaos are dense in the strange attractor [6]. If a specific chaotic trajectory is followed during the entire moment, the trajectory always remains arbitrarily close to closed orbits. Actually, the wandering of the trajectory from the neighbourhood of one closed orbit to another is one of the main characteristics of chaos. Suppose that a chaotic trajectory close to a closed orbit is analysed for a short period of time. The trajectory will share some of the same properties as the closed orbit. For example, if over a short duration a spectral analysis is carried out, they probably have very similar spectra. So the phase relationship between different frequencies for the closed orbit in this short time is shared with that of the actual trajectory. Suppose that the trajectory approaches two closed orbits, as in Figure 1. Usually, the properties of the trajectory near the closed orbit 1 are not the same as the properties of the trajectory near the closed orbit 2. But if some properties of the two closed orbits are invariant, they could be observed through the analysis of the trajectory. One of the main issues in this paper is to study the invariant properties of the closed orbits during chaos. It has been found that the closed orbits can be divided into groups according to these invariant properties.

In what follows, a theory related to closed orbits in chaotic Duffing oscillators is presented. The closed orbits can be divided into many groups. Within each group there are at least two closed orbits. One closed orbit is just another closed orbit with π phase shift on some Fourier components. It was found that the π phase shift does not affect the coherence between ω and 3ω frequencies. But it affects the coherence between ω and 2ω. Using an alternative TSC which corrects the π shift, the coherence of ω and 2ω has been

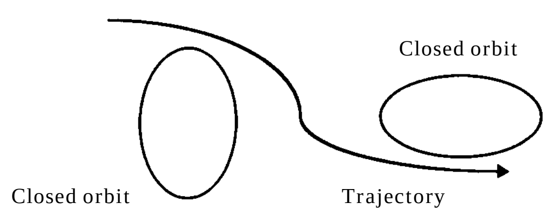

Figure 1. Illustration of two closed orbits. A chaotic trajectory wanders from the neighbourhood of one trajectory to another. (The closed orbits are highly simplified for the sake of illustration.)

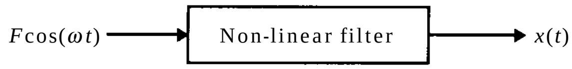

Figure 2. Illustration of the non-linear filter for Duffing oscillator (8).

revealed. It is also explained how the phase coherence between ω and the sub-harmonics, which exists in periodic case, disappeared during chaos.

2. BACKGROUND TO TRANS-SPECTRAL-COHERENCE

To illustrate the basic idea, consider a special case of two sets of signals at frequencies ω and 3ω. Each set consists of several samples of $x(t)$ or $y(t)$. Such sets can be represented as

$$x^{(m)}(t) = a^{(m)} \cos [(2\pi f)t + \varphi^{(m)}], \qquad m = 1, \ldots, N_{rec} \tag{1}$$

and

$$y^{(m)}(t) = b^{(m)} \cos [3(2\pi f)t + \theta^{(m)}], \qquad m = 1, \ldots, N_{rec}, \tag{2}$$

where N_{rec} represents the number of records and the superscript, (m), represents the index number of the record of $x(t)$ or $y(t)$ ranging from 1 to N_{rec}; the amplitudes and phases vary from one record to another. The trans-phase difference between $x(t)$ and $y(t)$ can be defined as

$$\tau^{(m)} = (3\varphi^{(m)} - \theta^{(m)}). \tag{3}$$

If $\tau^{(m)}$ remains constant over all the records, the signal $x(t)$ and $y(t)$ will be defined to be trans-spectrally coherent [4]. To measure coherence, calculate

$$C_1 = \langle \cos (\tau^{(m)}) \rangle_m = \langle \cos (3\varphi^{(m)} - \theta^{(m)}) >_m, \tag{4}$$

and

$$C_2 = \langle \sin (\tau^{(m)}) \rangle_m = \langle \sin 3\varphi^{(m)} - \theta^{(m)} \rangle_m. \tag{5}$$

Now, the TSC, in this case, is given by

$$\mathrm{TSC}(f, 3f) = \sqrt{C_1^2 + C_2^2}, \tag{6}$$

and the trans-phase is given by

$$\tau(f, 3f) = \tan^{-1}\left(\frac{C_2}{C_1}\right). \tag{7}$$

For a more general definition of the TSC, please refer to reference [4]. However, this discussion should be sufficient to grasp most of the ideas which follow.

3. TRANS-SPECTRAL-COHERENCE IN THE DUFFING OSCILLATOR

Now, let us discuss the results obtained by applying the TSC technique in the following Duffing oscillator:

$$\ddot{x} + c\dot{x} + kx + \delta x^3 = F \cos (\omega t + \alpha). \tag{8}$$

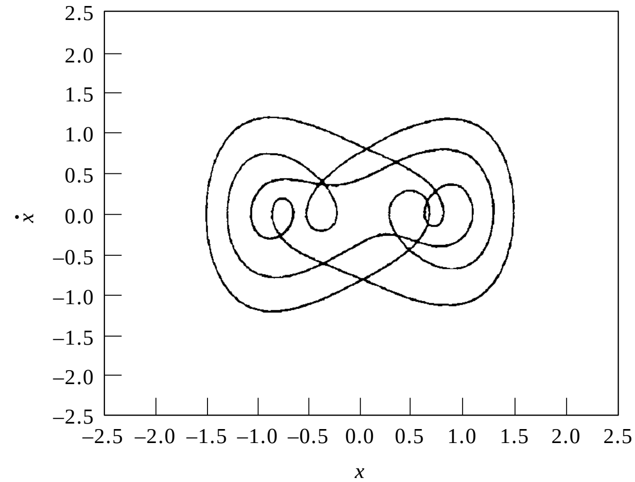

Figure 3. The phase portrait of the Duffing oscillator (8). The parameters are: $c = 0{\cdot}044964$, $k = 0$, $\delta = 1$, $\omega = 0{\cdot}44964$ and $F = 0{\cdot}7$.

One way to look at the Duffing oscillator is by the metaphor of a non-linear filter: the input term is $F \cos(\omega t + \alpha)$; the output is the solution $x(t)$, as Figure 2 shows. The Trans-Spectral-Coherence (TSC) technique can be used to analyse the coherence between the input and output. Consider the TSC between the specific frequency $f = \omega/(2\pi)$ of the input $F \cos(\omega t + \alpha)$ and each frequency of the output $x(t)$. This type of the TSC is referred to as the "sweeping TSC".

Let the parameters be $c = 0{\cdot}044964$, $k = 0$, $\delta = 1$, $\omega = 0{\cdot}44964$ and $\alpha = 0{\cdot}5$, as related to the parameters used by Ueda [1] and Rahman and Burton [7]. Then the equation is

$$\ddot{x} + 0{\cdot}044964\dot{x} + x^3 = F \cos(0{\cdot}44964t + 0{\cdot}5). \tag{9}$$

When $F = 0{\cdot}7$, the solution of equation (9) is periodic. Its limit cycle can be seen in Figure 3. The power spectrum and TSC spectrum are in Figures 4 and 5, respectively. In

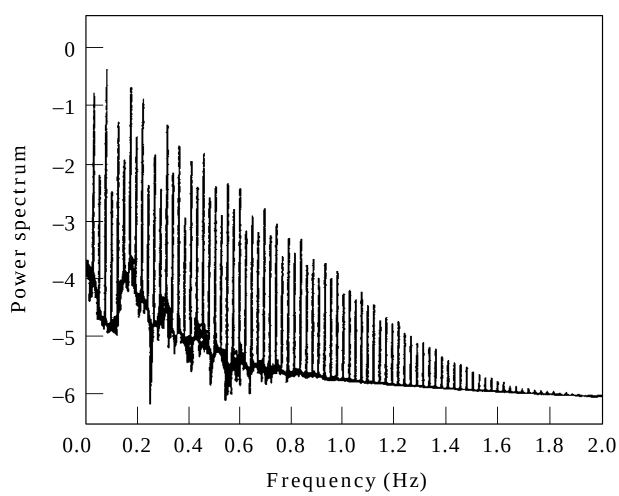

Figure 4. The power spectrum of the solution $x(t)$ in the Duffing oscillator (8). The parameters are: $c = 0{\cdot}044964$, $k = 0$, $\delta = 1$, $\omega = 0{\cdot}44964$ and $F = 0{\cdot}7$.

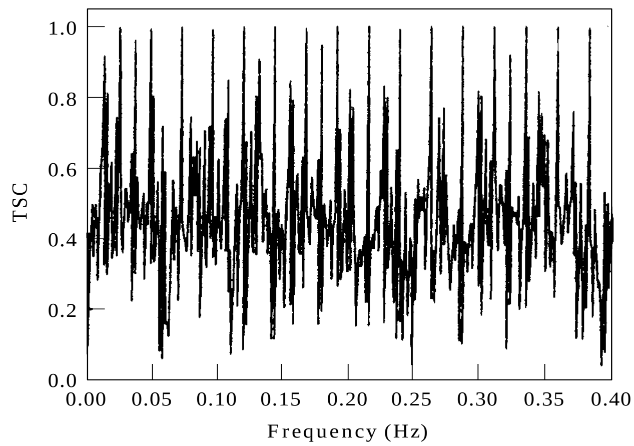

Figure 5. The sweeping TSC spectrum between the input $F\cos(\omega t + \alpha)$ and each frequency of $x(t)$ in the Duffing oscillator (8). The parameters are: $c = 0{\cdot}044964$, $k = 0$, $\delta = 1$, $\omega = 0{\cdot}44964$, $\alpha = 0{\cdot}5$ and $F = 0{\cdot}7$.

the calculation, there are in total $N_{rec} = 200$ records and there are 10,000 data in each record. The sampling interval time is $12\pi/(1000\omega)$ s.

In Figure 5, the peaks show that the Fourier components of the output $x(t)$ have the phase coherence with frequency $\omega/(2\pi) = 0{\cdot}0716$ Hz of the input signal. As expected based on findings in earlier work [3], it can be seen that there is very strong phase coherence between the frequency ω of the input and the frequencies

$$\tfrac{1}{3}\omega, \tfrac{2}{3}\omega, \omega, \tfrac{4}{3}\omega, \tfrac{5}{3}\omega, 2\omega, \ldots, \text{etc.}$$

of the output.

If $F = 1{\cdot}05$, the Duffing oscillator is chaotic [7]. The phase portrait can be seen in Figure 6. The power spectrum of $x(t)$ is shown in Figure 7 and the sweeping TSC spectrum is shown in Figure 8.

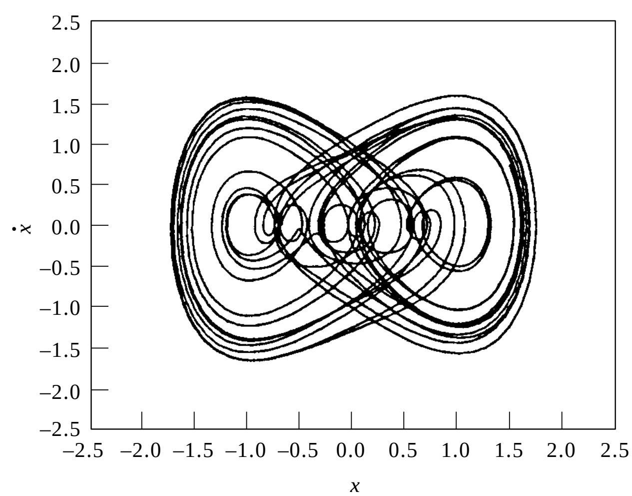

Figure 6. The phase portrait of the Duffing oscillator (8). The parameters are: $c = 0{\cdot}044964$, $k = 0$, $\delta = 1$, $\omega = 0{\cdot}44964$, $\alpha = 0{\cdot}5$ and $F = 1{\cdot}05$.

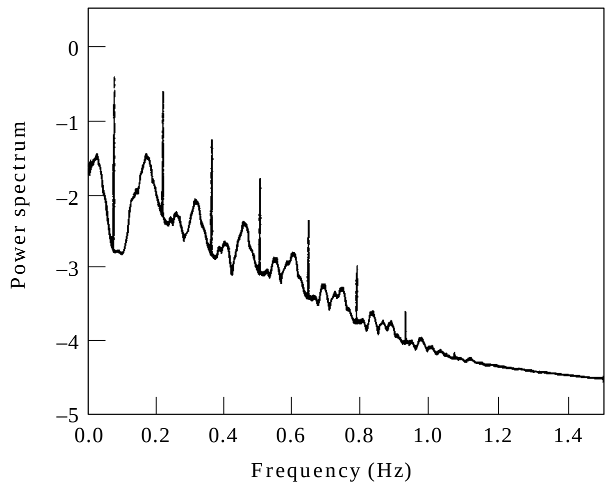

Figure 7. The power spectrum of the solution $x(t)$ in the Duffing oscillator (8). The parameters are: $c = 0{\cdot}044964$, $k = 0$, $\delta = 1$, $\omega = 0{\cdot}44964$, $\alpha = 0{\cdot}5$ and $F = 1{\cdot}05$.

Figure 8 shows that, as expected, the phase coherence between the frequency ω of the input and the frequencies 2ω, 4ω, . . . , etc. of the output has vanished. However, in spite of chaos, there still exists the phase coherence in chaos. There is very strong phase coherence between the frequency ω of the input and the frequencies ω, 3ω, 5ω, . . . , etc. of the output.

The results were quite surprising so we decided to check if they remained true for other parametric values. Now we chose the ones used by Ueda [1] and Guckenheimer and Holmes [2]. The parameters are $c = 0{\cdot}25$, $k = -1$, $\delta = 1$, $\omega = 1$ and $\alpha = 0{\cdot}5$. The equation is

$$\ddot{x} + 0{\cdot}25\dot{x} - x + x^3 = F\cos(t + 0{\cdot}5). \tag{10}$$

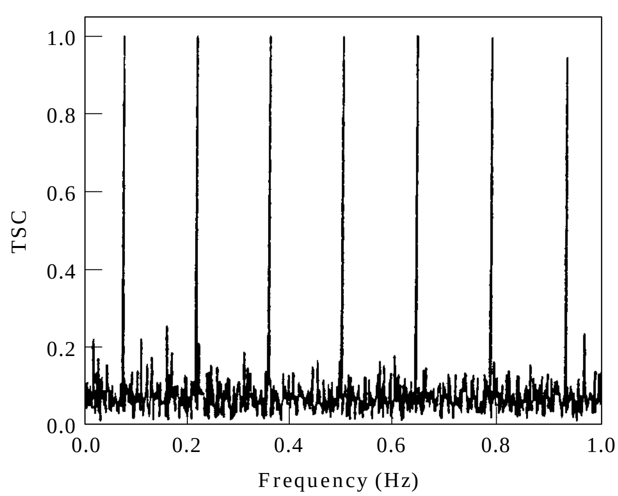

Figure 8. The sweeping TSC spectrum between the input $F\cos(\omega t + \alpha)$ and each frequency of $x(t)$ in the Duffing oscillator (8). The parameters are: $c = 0{\cdot}044964$, $k = 0$, $\delta = 1$, $\omega = 0{\cdot}44964$, $\alpha = 0{\cdot}5$ and $F = 1{\cdot}05$.

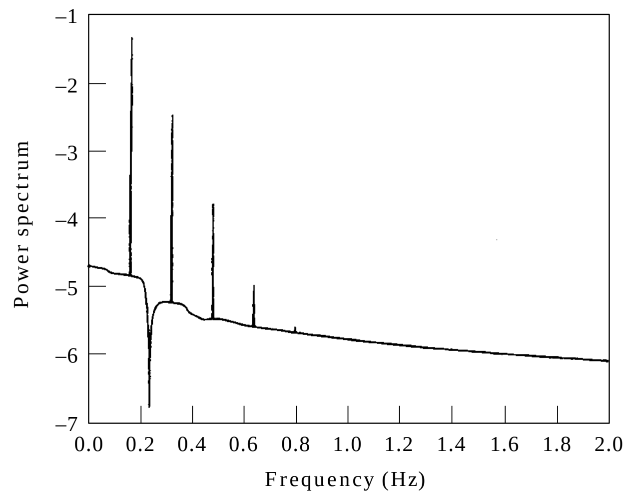

Figure 9. The power spectrum of the solution $x(t)$ in the Duffing oscillator (8). The parameters are: $c = 0{\cdot}25$, $k = -1{\cdot}0$, $\delta = 1{\cdot}0$, $\omega = 1{\cdot}0$, $\alpha = 0{\cdot}5$ and $F = 0{\cdot}1$.

The sampling interval time is $12\pi/1000$. The frequency of the input is $\omega/(2\pi) = 0{\cdot}1592$ Hz. If $F = 0{\cdot}1$, equation (10) is periodic. The power spectrum of the solution (or output) $x(t)$ is shown in Figure 9, and the TSC spectrum in Figure 10. If $F = 0{\cdot}30$, equation (10) is a well-known chaotic Duffing oscillator. The power spectrum of $x(t)$ is shown in Figure 11, and the TSC spectrum in Figure 12. These results are similar to previous results. A summary of these results is given in Table 1.

Understanding how this coherence persists and the change of its character is one of the central goals of this paper. Consider Figure 2 in which the input is $F\cos(\omega t + \alpha)$. The output is the solution $x(t)$ of the Duffing oscillator. If the output $x(t)$ is periodic, it can be written in the form of the Fourier series:

$$\text{Output} = x(t) = B_0 + B_1\cos(\omega_1 t + \varphi_1) + B_2\cos(\omega_2 t + \varphi_2) + B_3\cos(\omega_3 t + \varphi_3) + \cdots, \tag{11}$$

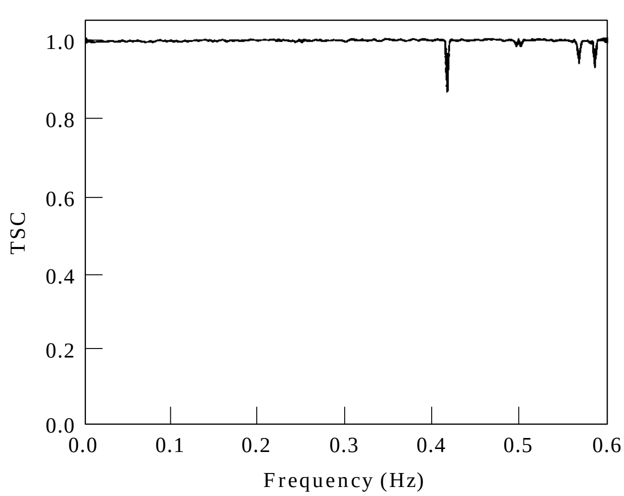

Figure 10. The sweeping TSC spectrum between the input $F\cos(\omega t + \alpha)$ and each frequency of $x(t)$ in the Duffing oscillator (8). The parameters are: $c = 0{\cdot}25$, $k = -1{\cdot}0$, $\omega = 1{\cdot}0$, $\alpha = 0{\cdot}5$ and $F = 0{\cdot}1$.

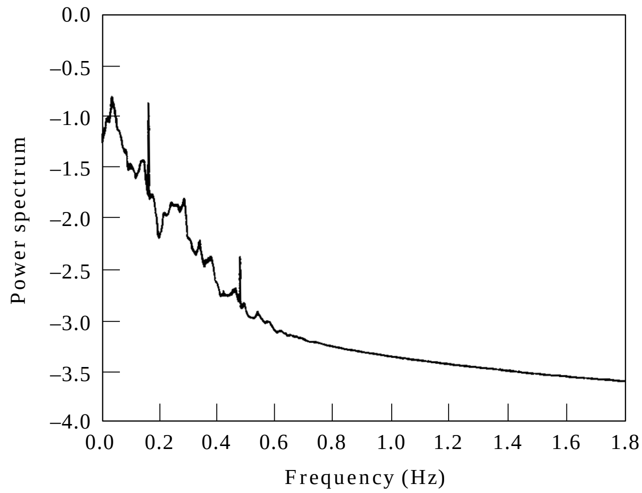

Figure 11. The power spectrum of $x(t)$ in the Duffing oscillator (8). The parameters are: $c = 0{\cdot}25$, $k = -1$, $\delta = 1$, $\omega = 1$ and $F = 0{\cdot}30$.

where B_0, B_1, B_2, ω_1, ω_2, ω_3, φ_1, φ_2, φ_3, etc. are all constants. Since the phases of the input and output are constants, they are totally coherent. The TSC between the input frequency ω and each of the output frequencies (i.e., $\omega_1, \omega_2, \ldots$) is exactly 1 [4].

However, in order to explain why there is strong coherence in chaos, the intrinsic properties of chaos need to be examined. These properties will be discussed in the following sections.

4. GROUP THEORY OF CLOSED ORBITS IN CHAOS

The closed orbits of chaos can be divided into many groups according to the Fourier series of the solution. The theory relating to these groups has been used to explain the

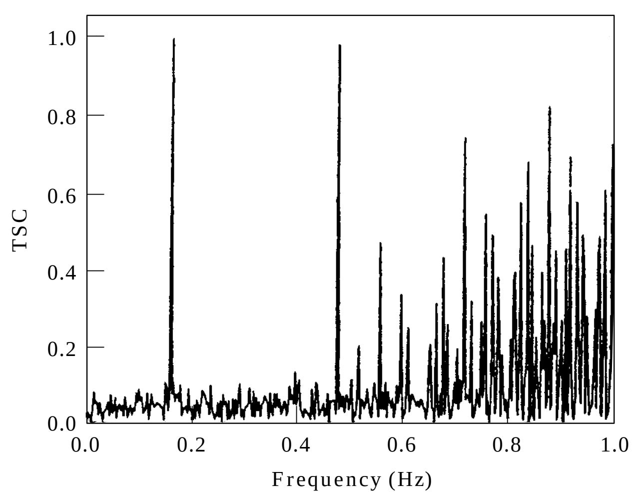

Figure 12. The sweeping TSC spectrum between the input $F\cos(\omega t + \alpha)$ and each frequency of $x(t)$ in the Duffing oscillator (8). The parameters are: $c = 0{\cdot}25$, $k = -1{\cdot}0$, $\delta = 1{\cdot}0$, $\omega = 1$, $\alpha = 0{\cdot}5$ and $F = 0{\cdot}3$.

TABLE 1

Trans-Spectral-Coherence of the Duffing oscillator

	Lyapunov exponents	Input ω and output ω, 3ω, 5ω, etc.	Input ω and output 2ω, 4ω, etc.
Equation (9) $F = 0{\cdot}7$	$(-0{\cdot}032, -0{\cdot}032)$	Coherence	Coherence
Equation (9) $F = 1{\cdot}05$	$(+0{\cdot}087, -0{\cdot}15)$	Coherence	Coherence not found
Equation (10) $F = 0{\cdot}10$	$(-0{\cdot}18, -0{\cdot}18)$	Coherence	Coherence
Equation (10) $F = 0{\cdot}30$	$(+0{\cdot}18, -0{\cdot}54)$	Coherence	Coherence not found

phase coherence in chaos. In the following, the closed orbits with only harmonics will first be discussed, then the closed orbits with sub-harmonics will be examined.

4.1. CLOSED ORBITS WITH HARMONICS

There are many closed orbits in chaos. Curry's algorithm [8] can be used to find the closed orbits embedded in the chaotic trajectory of the Duffing oscillator. Consider an example:

$$\ddot{x} + 0{\cdot}25\dot{x} - x + x^3 = 0{\cdot}30 \cos(t). \tag{12}$$

Equation (12) is chaotic. One of the closed orbits found is shown in Figure 13. It is on the left side of the phase plane. The co-ordinate of the initial point $(x, \dot{x})$ at $t = 0$ on this orbit is $(-1{\cdot}10194568415012184, 0{\cdot}636462338217809226)$ and the period is $T = 2\pi$. (Note: the IBM3090 computer and 5th order Runge-Kutta method in the IMSL package were

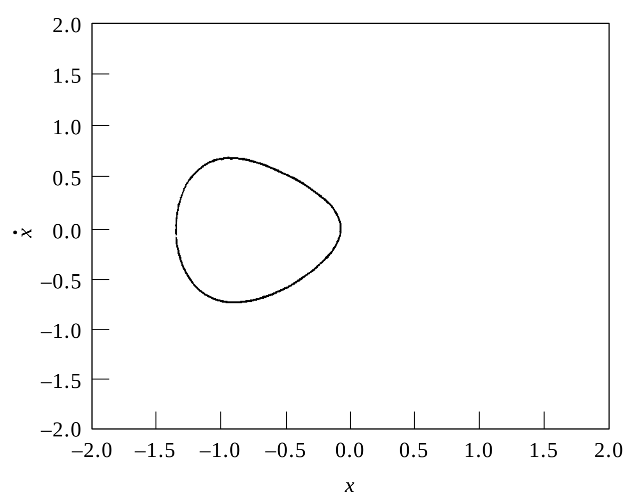

Figure 13. One closed orbit in the Duffing oscillator (12). The period is $T = 2\pi$.

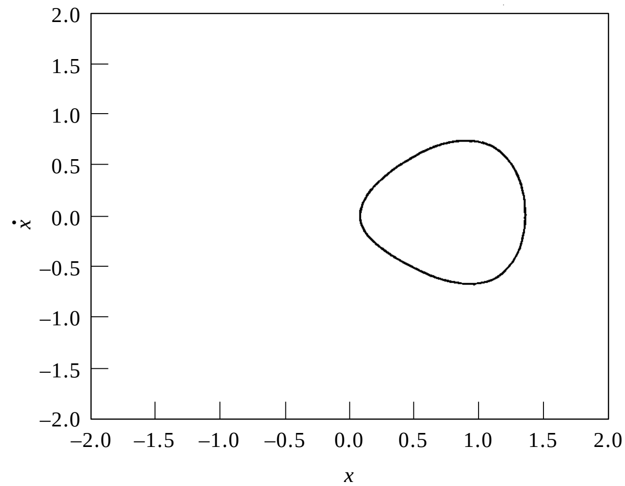

Figure 14. Adjoint of the closed orbit in the Duffing oscillator (12). The period is $T = 2\pi$.

used). Let $x_l(t)$ represent the function of this limit cycle. From the power spectra, $x_l(t)$ can be written in the form of the Fourier series:

$$\begin{aligned} x_l(t) \approx &-0{\cdot}602197111 + 0{\cdot}61325\cos(t + 3{\cdot}7637) + 0{\cdot}1093022\cos(2t + 4{\cdot}55047) \\ &+ 0{\cdot}022663\cos(3t + 5{\cdot}22054) + 0{\cdot}00456384\cos(4t + 5{\cdot}91212) \\ &+ 0{\cdot}000458186\cos(5t + 0{\cdot}31943) + 0{\cdot}000919227\cos(6t + 1{\cdot}01108) + \cdots. \end{aligned} \tag{13}$$

Note: the leading term, $-0{\cdot}602197111$, is obtained by averaging $x(t)$ values for one period. There is another closed orbit on the right side of the phase plane, shown in Figure 14. Obviously, there exists symmetry between Figures 13 and 14. The co-ordinate of the initial point $(x, \dot{x})$ at $t = 0$ on this orbit is (0·129005596571571526, 0·193679445874795686) and the period is $T = 2\pi$. Let $x_{\bar{l}}(t)$ represent the function of this limit cycle. It can be written as

$$\begin{aligned} x_{\bar{l}}(t) \approx\ &0{\cdot}602197289 + 0{\cdot}61325\cos(t + 3{\cdot}7637) + 0{\cdot}1093022\cos(2t + 4{\cdot}55047 + \pi) \\ &+ 0{\cdot}022663\cos(3t + 5{\cdot}22054) + 0{\cdot}00456384\cos(4t + 5{\cdot}91212 + \pi) \\ &+ 0{\cdot}000458186\cos(5t + 0{\cdot}31943) + 0{\cdot}000919227\cos(6t + 1{\cdot}01108 + \pi) + \cdots. \end{aligned} \tag{14}$$

This specific observation can be generalized. Without loss of generality, let $x_i(t)$ represent one of the limit cycles. For the sake of simplicity, consider only those limit cycles without sub-harmonic components. Thus, it is assumed that $x_i(t)$ has the form

$$x_i(t) = B_0 + B_1\cos(\omega t + \varphi_1) + B_2\cos(2\omega t + \varphi_2) + B_3\cos(3\omega t + \varphi_3) + \cdots, \tag{15}$$

where B_0, B_1, B_2, B_3, φ_1, φ_2, φ_3, etc. may not be constants. $x_i(t)$ is divided into two groups:

$$x_i(t) = D_i(t) + E_i(t), \tag{16}$$

where

$$D_i(t) = B_1\cos(\omega t + \varphi_1) + B_3\cos(3\omega t + \varphi_3) + \cdots, \tag{17}$$

and

$$E_i(t) = B_0 + B_2 \cos(2\omega t + \varphi_2) + B_4 \cos(4\omega t + \varphi_4) + \cdots . \tag{18}$$

The components with any of the frequencies $2n\omega$ (where $n = 0, 1, 2, \ldots$) are referred to as "the components of even frequencies" and the components with any of the frequencies $(2n + 1)\omega$ (where $n = 0, 1, 2, \ldots$) as "the components of odd frequencies".

In equation (13), one has

$$D_l(t) \approx 0{\cdot}61325 \cos(t + 3{\cdot}7637) + 0{\cdot}022663 \cos(3t + 5{\cdot}22054) + 0{\cdot}000458186 \cos(5t + 0{\cdot}31943) + \cdots , \tag{19}$$

and

$$E_l(t) \approx -0{\cdot}602197111 + 0{\cdot}1093022 \cos(2t + 4{\cdot}55047) + 0{\cdot}00456384 \cos(4t + 5{\cdot}91212) + 0{\cdot}000919227 \cos(6t + 1{\cdot}0118) + \cdots . \tag{20}$$

For $x_{\bar{l}}(t)$, one has

$$D_{\bar{l}}(t) \approx 0{\cdot}61325 \cos(t + 3{\cdot}7637) + 0{\cdot}022663 \cos(3t + 5{\cdot}22054) + 0{\cdot}000458186 \cos(5t + 0{\cdot}31943) + \cdots ; \tag{21}$$

$$E_{\bar{l}}(t) \approx 0{\cdot}602197289 + 0{\cdot}1093022 \cos(2t + 4{\cdot}55047 + \pi) + 0{\cdot}00456384 \cos(4t + 5{\cdot}91212 + \pi) + 0{\cdot}000919227 \cos(6t + 1{\cdot}01108 + \pi) + \cdots \tag{22}$$

Comparing D_l, E_l and $D_{\bar{l}}$, $E_{\bar{l}}$, it can be seen that $D_l = D_{\bar{l}}$, and $E_{\bar{l}}$ is just E_l with π phase shift (B_0 can be considered as $B_0 \cos(0\omega)$). Then,

$$E_l = -E_{\bar{l}}. \tag{23}$$

Clearly, this is in agreement with the following theorem which is proved in Appendix A.

Theorem: If the Duffing oscillator

$$\ddot{x} + c\dot{x} + kx + \delta x^3 = F \cos(\omega t + \alpha) \tag{24}$$

is chaotic and $x_i(t) = D_i(t) + E_i(t)$ is one of the closed orbits, then $x_{\bar{i}} = D_i(t) - E_i(t)$ is another closed orbit, where

$$D_i(t) = B_1 \cos(\omega t + \varphi_1) + B_3 \cos(3\omega t + \varphi_3) + \cdots , \tag{25}$$

$$E_i(t) = B_0 + B_2 \cos(2\omega t + \varphi_2) + B_4 \cos(4\omega t + \varphi_4) + \cdots . \tag{26}$$

Actually, the above theorem can be extended to some symmetric systems. Suppose that a system is $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, where $\mathbf{x}$ and $\mathbf{f}$ are vectors. If the system has the symmetry such that

$$\mathbf{f}(\mathbf{x}, t) = -\mathbf{f}\left(-\mathbf{x}, t + \frac{T}{2}\right), \tag{27}$$

where T is the period of a periodic solution $x(t)$, then the periodic solution has the symmetry:

$$\mathbf{x}(t) = -\mathbf{x}\left(t + \frac{T}{2}\right). \tag{28}$$

It is easy to see that this kind of symmetric solution satisfies the above theorem. Note that the sine-Gordon equation, studied in reference [3], has the same symmetry.

4.2. CLOSED ORBITS WITH SUB-HARMONICS

Now, consider the closed orbits with sub-harmonics. Let $x_s(t)$ be one closed orbit with sub-harmonics. $x_s(t)$ can be written as

$$x_s(t) = B_0 + B_1 \cos(\Omega t + \varphi_1) + B_2 \cos(2\Omega t + \varphi_2) + B_3 \cos(3\Omega t + \varphi_3) + \cdots, \quad (29)$$

where $\Omega = \omega/m$ and m is a positive integer. The case $m = 1$ will not be considered, because it is the same case as in section 4.1. If $m > 1$, there exist the sub-harmonics in $x_s(t)$. Similar to section 4.1, if the components of odd frequencies are defined as

$$D_s(t) = B_1 \cos(\Omega t + \varphi_1) + B_3 \cos(3\Omega t + \varphi_3) + B_5 \cos(5\Omega t + \varphi_5) + \cdots, \quad (30)$$

and the components of even frequencies as

$$E_s(t) = B_0 + B_2 \cos(2\Omega t + \varphi_2) + B_4 \cos(4\Omega t + \varphi_4) + \cdots, \quad (31)$$

the results are similar to those in the last section. As an example, Figure 15 is one of the closed orbits of equation (12) with $\omega/3$ sub-harmonics. This orbit starts from the initial condition $(x, \dot{x}) = (0{\cdot}28934598812256953, 0{\cdot}399977592441817215)$ at $t = 0$ and the period is $T = 3 \times 2\pi$. Its Fourier series is:

$$\begin{aligned} x_s(t) \approx\ & -0{\cdot}202490 + 0{\cdot}92834 \cos(\tfrac{1}{3}\omega t + 6{\cdot}1863) + 0{\cdot}29418 \cos(\tfrac{2}{3}\omega t + 4{\cdot}539805) \\ & + 0{\cdot}38398 \cos(\tfrac{3}{3}\omega t + 4{\cdot}06990) + 0{\cdot}071126 \cos(\tfrac{4}{3}\omega t + 2{\cdot}10259) \\ & + 0{\cdot}13408 \cos(\tfrac{5}{3}\omega t + 3{\cdot}93422) + 0{\cdot}070459 \cos(\tfrac{6}{3}\omega t + 2{\cdot}09298) \\ & + 0{\cdot}027621 \cos(\tfrac{7}{3}\omega t + 1{\cdot}99053) + 0{\cdot}015391 \cos(\tfrac{8}{3}\omega t + 0{\cdot}57096) \\ & + 0{\cdot}011202 \cos(\tfrac{9}{3}\omega t + 1{\cdot}46067) + 0{\cdot}010539 \cos(\tfrac{10}{3}\omega t + 6{\cdot}05879) \\ & + 0{\cdot}00025698 \cos(\tfrac{11}{3}\omega t + 5{\cdot}50328) + 0{\cdot}00025182 \cos(\tfrac{12}{3}\omega t + 4{\cdot}80375) \\ & + 0{\cdot}00091633 \cos(\tfrac{13}{3}\omega t + 4{\cdot}82651) + 0{\cdot}0013183 \cos(\tfrac{14}{3}\omega t + 3{\cdot}75092) \\ & + 0{\cdot}00040081 \cos(\tfrac{15}{3}\omega t + 2{\cdot}67029) + 0{\cdot}0032773 \cos(\tfrac{16}{3}\omega t + 2{\cdot}55312) \\ & + 0{\cdot}00014206 \cos(\tfrac{17}{3}\omega t + 1{\cdot}77279) + 0{\cdot}00014688 \cos(\tfrac{18}{3}\omega t + 1{\cdot}36668) + \cdots \end{aligned} \quad (32)$$

Figure 16 is another closed orbit of equation (12) with $\omega/3$ sub-harmonics. This orbit starts from the initial condition $(x, \dot{x}) = (0{\cdot}8934453767438, 0{\cdot}426445223559)$ at $t = 0$ and the period is $T = 3 \times 2\pi$. Its Fourier series is:

$$\begin{aligned} x_{\bar{s}}(t) \approx\ & 0{\cdot}202487 + 0{\cdot}92834 \cos(\tfrac{1}{3}\omega t + 6{\cdot}1863) + 0{\cdot}29418 \cos(\tfrac{2}{3}\omega t + 4{\cdot}539805 + \pi) \\ & + 0{\cdot}38398 \cos(\tfrac{3}{3}\omega t + 4{\cdot}06990) + 0{\cdot}071126 \cos(\tfrac{4}{3}\omega t + 2{\cdot}10259 + \pi) \\ & + 0{\cdot}13408 \cos(\tfrac{5}{2}\omega t + 3{\cdot}93422) + 0{\cdot}070459 \cos(\tfrac{6}{3}\omega t + 2{\cdot}09298 + \pi) \\ & + 0{\cdot}027621 \cos(\tfrac{7}{3}\omega t + 1{\cdot}99053) + 0{\cdot}015391 \cos(\tfrac{8}{3}\omega t + 0{\cdot}57096 + \pi) \\ & + 0{\cdot}011202 \cos(\tfrac{9}{3}\omega t + 1{\cdot}46067) + 0{\cdot}010539 \cos(\tfrac{10}{3}\omega t + 6{\cdot}05879 + \pi) \\ & + 0{\cdot}00025698 \cos(\tfrac{11}{3}\omega t + 5{\cdot}50328) + 0{\cdot}00025182 \cos(\tfrac{12}{3}\omega t + 4{\cdot}80375 + \pi) \\ & + 0{\cdot}00091633 \cos(\tfrac{13}{3}\omega t + 4{\cdot}82651) + 0{\cdot}0013183 \cos(\tfrac{14}{3}\omega t + 3{\cdot}75092 + \pi) \\ & + 0{\cdot}00040081 \cos(\tfrac{15}{3}\omega t + 2{\cdot}67029) + 0{\cdot}00032773 \cos(\tfrac{16}{3}\omega t + 2{\cdot}55312 + \pi) \\ & + 0{\cdot}00014206 \cos(\tfrac{17}{3}\omega t + 1{\cdot}77279) + 0{\cdot}00014688 \cos(\tfrac{18}{3}\omega t + 1{\cdot}3668 + \pi) \\ & + \cdots . \end{aligned} \quad (33)$$

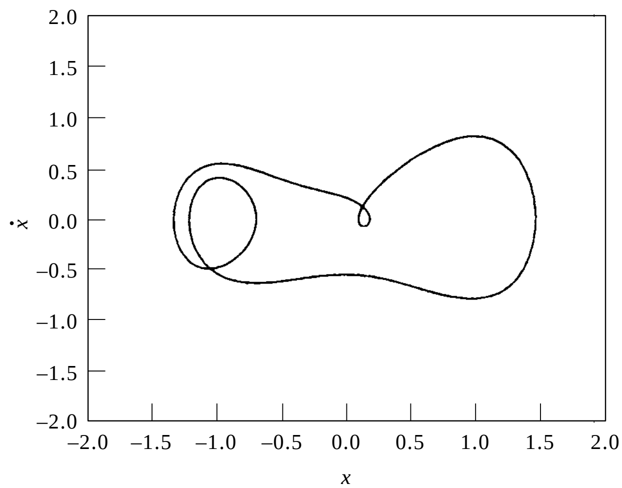

Figure 15. One closed orbit in the Duffing oscillator (12). The period is $T = 3 \times 2\pi$.

One can see that $x_s(t)$ equals $x_{\bar{s}}(t)$ with π phase shift on the components of even frequencies.

For a closed orbit, not every point of the orbit can be an initial condition which can lead to this closed orbit. Without loss of generality, it is assumed that the initial condition is at $t = 0$. However, there are many initial conditions which can lead to the same closed orbit. For example, consider a chaotic equation:

$$\ddot{x} + 0{\cdot}25\dot{x} - x + x^3 = 0{\cdot}3 \cos(t). \tag{34}$$

By Curry's algorithm [8], a close orbit with $\omega/3$ sub-harmonics can be found, as Figure 15 shows. This orbit starts from the initial condition $(x, \dot{x}) = (0{\cdot}2893459881226, 0{\cdot}39997759244182)$ at $t = 0$ and the period is $T = 3 \times 2\pi$. However, the initial condition

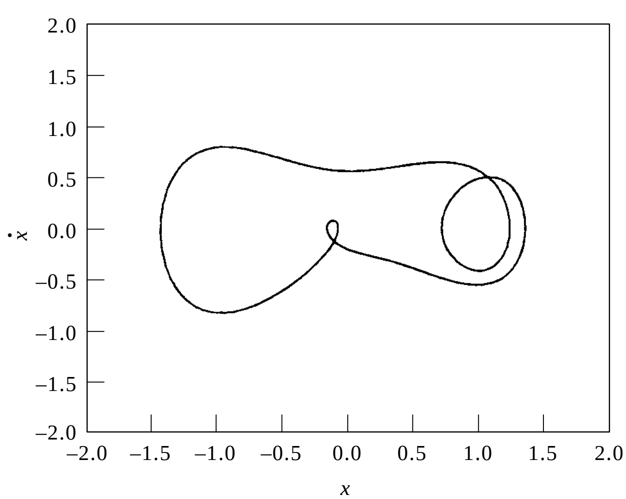

Figure 16. Adjoint of the closed orbit in the Duffing oscillator (12). The period is $T = 3 \times 2\pi$.

$(x, \dot{x}) = (-0{\cdot}5214372386756,\ 0{\cdot}40558755678792)$ at $t = 0$, is chosen the same phase portrait as in Figure 15 can be obtained. Although starting from these two initial conditions one has the same two closed orbits, their trajectories are different, i.e., their solutions have different time history. If the Fourier series is used to represent these closed orbits, they will have different phase angles.

Theorem: (proved in Appendix B) For Duffing oscillator

$$\ddot{x} + c\dot{x} + kx + \delta x^3 = F\cos(\omega t), \tag{35}$$

if

$$x_0 = \sum_{n=0}^{\infty} B_n \cos(n\Omega t + \varphi_n) \tag{36}$$

is one solution, then

$$x_L = \sum_{n=0}^{\infty} B_n \cos\left(n\Omega t' + \varphi_n - \frac{2\pi n L}{m} + (m+n)\pi\right) \tag{37}$$

are also solutions, where $m\Omega = \omega$; m is a positive integer and $L = 0, 1, 2, \ldots, m-1$.

As an example, for Figure 15, if this orbit starts from the initial condition $(x, \dot{x}) = (0{\cdot}2893459881226, 0{\cdot}39997759244182)$ at $t = 0$ and the period is $T = 3 \times 2\pi$, then the Fourier series is:

$$\begin{aligned} x_0(t) \approx\ & -0{\cdot}202490 + 0{\cdot}92834\cos(\tfrac{1}{3}\omega t + 6{\cdot}1863) + 0{\cdot}29418\cos(\tfrac{2}{3}\omega t + 4{\cdot}539805) \\ & + 0{\cdot}38398\cos(\omega t + 4{\cdot}06990) + 0{\cdot}071126\cos(\tfrac{4}{3}\omega t + 2{\cdot}10259) \\ & + 0{\cdot}13408\cos(\tfrac{5}{3}\omega t + 3{\cdot}93422) + 0{\cdot}070459\cos(2\omega t + 2{\cdot}09298) + \cdots . \end{aligned} \tag{38}$$

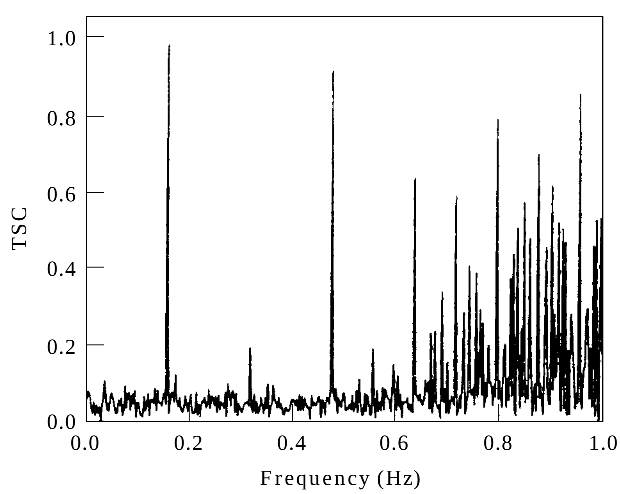

Figure 17. The sweeping TSC spectrum (with the phase angles multiplied by 2) between the input $F\cos(\omega t + \alpha)$ and each frequency of $x(t)$ in the Duffing oscillator (8). The parameters are: $c = 0{\cdot}25$, $k = -1$, $\delta = 1$, $\omega = 1$, $\alpha = 0{\cdot}5$ and $F = 0{\cdot}3$.

If the initial condition $(x, \dot{x}) = (-0{\cdot}5214372386756, 0{\cdot}40558755678792)$ at $t = 0$ is chosen, the Fourier series is

$$x_0(t) \approx -0{\cdot}202490 + 0{\cdot}92834\cos(\tfrac{1}{3}\omega t + 4{\cdot}0919) + 0{\cdot}29418\cos(\tfrac{2}{3}\omega t + 0{\cdot}351014)$$

$$+ 0{\cdot}38398\cos(\omega t + 4{\cdot}06990) + 0{\cdot}071126\cos(\tfrac{4}{3}\omega t + 0{\cdot}0081948)$$

$$+ 0{\cdot}13408\cos(\tfrac{5}{3}\omega t + 6{\cdot}028615) + 0{\cdot}070459\cos(2\omega t + 2{\cdot}09298) + \cdots . \quad (39)$$

Equations (38) and (39) satisfy the above theorem.

5. EXPLANATION OF THE TRANS-SPECTRAL-COHERENCE

From the analysis in the previous sections, it is known that there are many closed orbits when the solutions of the Duffing equation are chaotic. These closed orbits can be divided into many groups. One group is the closed orbits without sub-harmonics; other groups are the closed orbits with sub-harmonics. The closed orbits with sub-harmonics are divided into the groups according to m, where $m\Omega = \omega$ (see section 4.2). Within each group, there are at least two closed orbits. One is $x_l = D + E$; another is $x_{\bar{l}} = D - E$ (or $D' - E' = +E + D$; see section 4.2). One closed orbit is another closed orbit with π phase shift in the components of even frequencies. This gives rise to an "adjoined" orbit for each closed orbits; as shown in Figures 13–16. Actually, according to the theorem in the last section, each group can have $2m$ closed orbits: m closed orbits and their adjoined closed orbits.

During chaos the actual trajectory never really coincides with any of the closed orbits. However, the orbits do come arbitrarily close to such orbits. The true trajectory will wander from the neighbourhood of one closed orbit to the neighbourhood of another closed orbit. However, within each group, the phase angles of the components of the frequencies ω, 3ω, 5ω, . . . , etc. will not change and the phase angles of the components of the frequencies 0ω, 2ω, 4ω, . . . , etc. will have π phase shift. This is why the strong TSC was obtained between the input signal which has ω frequency component and those $\omega, 3\omega, 5\omega, \ldots$, etc. components of the output signal and the TSC was not obtained between the input and the $2\omega, 4\omega, \ldots$, etc. components of the output signal.

Based on the above analysis, one should find the coherence between the input and the frequencies $2\omega, 4\omega, \ldots$, etc. of the output if the π phase shift is corrected. Now a special method is used to calculate the TSC: each phase angle is multiplied by 2 in TSC calculation. Thus, the π phase shift will become 2π and does not affect the TSC, because $\cos(\varphi + 2\pi) = \cos(\varphi)$. In this way, the TSC can be obtained between the input and the $2\omega, 4\omega, \ldots$, etc. components of the output signal, as Figures 17 and 18 show.

In chaos, the trajectory is moving around from the neighbourhood of one closed orbit to the neighbourhood of another closed orbit. For a closed orbit with sub-harmonics (i.e., $m > 1$), according to the above theorem, it has many expressions in the form of the Fourier series. Among these expressions, the phase angles of the sub-harmonics are different. So each time the trajectory can approach this closed orbit with the different phase angles of the sub-harmonics. Of these different phase angles, which phase angle is chosen each time can be random. For the harmonics, the trajectory approaches the closed orbit with the same phase angles each time, also according to the above theorem. Therefore, it can be concluded that in chaos there does not exist the phase coherence between the fundamental frequency ω of the input and the sub-harmonics of the output.

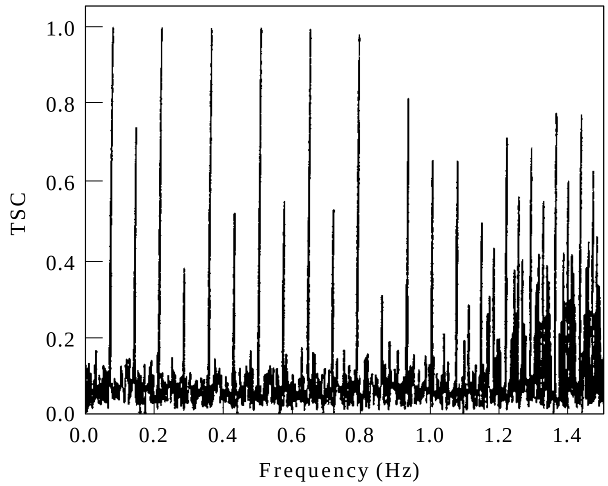

Figure 18. The sweeping TSC spectrum (with phase angles multiplied by 2) between each frequency of $x(t)$ and the input $F\cos(\omega t)$. In the Duffing oscillator (8), the parameters are: $c = 0{\cdot}044964$, $k = 0$, $\delta = 1$, $\omega = 0{\cdot}44964$ and $F = 1{\cdot}05$.

6. CONCLUSION

In this paper, the Trans-Spectral-Coherence technique has been used to analyse the phase coherence of the Duffing oscillator in the periodic and chaotic cases. By the TSC technique, strong phase coherence has been detected in the periodic case. In the chaotic case, the phase coherence between ω and $2\omega, 4\omega, \ldots,$ etc. vanishes. However, the strong coherence between ω and $3\omega, 5\omega, \ldots,$ etc. has been detected.

The closed orbits of chaos and their effect on the phase coherence have been studied. The closed orbits of chaos are dense in the strange attractor. The chaotic trajectory always remains arbitrarily close to some closed orbit or the other. Actually, the trajectory wanders from the neighbourhood of one closed orbit to another. As the reference point along the chaotic trajectory wanders from the neighbourhood of one closed orbit to another, if the closed orbits share common trans-phases, the chaotic trajectory may display trans-phase coherence, in spite of its wandering character. The closed orbits have been divided into many groups. Within each group there are at least two closed orbits whose phase portraits are symmetric due to the π phase shift on some of their Fourier components. The π phase shift does not affect the phase coherence between the fundamental frequency ω and the harmonics $3\omega, 5\omega, \ldots,$ etc. However, it affects the phase coherence between ω and $2\omega, 4\omega, \ldots,$ etc. harmonics. After correcting the phase shift, the coherence between ω and 2ω has been found. In addition, it has been shown that the phase coherence does not exist between the fundamental frequency ω and the sub-harmonics of the solution. The results in this paper also showed that the TSC technique is a powerful and efficient method in the analysis of non-linear signals.

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APPENDIX A: GROUP THEORY FOR HARMONICS

Let $x_i(t)$ represent one of the limit cycles. For now, consider only those limit cycles without sub-harmonic components. Thus, assume that $x_i(t)$ has the form

$$x_i(t) = B_0 + B_1 \cos(\omega t + \varphi_1) + B_2 \cos(2\omega t + \varphi_2) + B_3 \cos(3\omega t + \varphi_3) + \cdots, \quad \text{(A1)}$$

where $B_0, B_1, B_2, B_3, \varphi_1, \varphi_2, \varphi_3$, etc. may not be constants. Divide $x_i(t)$ into two groups:

$$x_i(t) = D_i(t) + E_i(t), \quad \text{(A2)}$$

where

$$D_i(t) = B_1 \cos(\omega t + \varphi_1) + B_3 \cos(3\omega t + \varphi_3) + \cdots, \quad \text{(A3)}$$

and

$$E_i(t) = B_0 + B_2 \cos(2\omega t + \varphi_2) + B_4 \cos(4\omega t + \varphi_4) + \cdots. \quad \text{(A4)}$$

Substituting equations (A3) and (A4) into equation (8), one has

$$\ddot{D}_i + c\dot{D}_i + kD_i + \ddot{E}_i + c\dot{E}_i + kE_i + \delta(D_i^3 + 3D_i^2E_i + 3D_iE_i^2 + E_i^3) = F\cos(\omega t + \alpha). \quad \text{(A5)}$$

The components with any of the frequencies $2n\omega$ (where $n = 0, 1, 2, \ldots$) are called "the components of even frequencies" and the components with any of the frequencies $(2n+1)\omega$ (where $n = 0, 1, 2, \ldots$), "the components of odd frequencies". Obviously, $\ddot{D}_i + c\dot{D}_i + kD_i$ belongs to "the components of odd frequencies" and $\ddot{E}_i + c\dot{E}_i + kE_i$ belongs to "the components of even frequencies". Now, consider D_i^3 and $D_iE_i^2$:

$$\begin{aligned} D_i^3 &= (B_1\cos(\omega t + \varphi_1) + B_3\cos(3\omega t + \varphi_3) + \cdots)^3 \\ &= B_1^3\cos^3(\omega t + \varphi_1) + B_3^3\cos^3(3\omega t + \varphi_3) + \cdots \\ &\quad + 3B_1^2B_3\cos^2(\omega t + \varphi_1)\cos(3\omega t + \varphi_3) + 3B_1^2B_5\cos^2(\omega t + \varphi_1)\cos(5\omega t + \varphi_5) + \cdots \\ &\quad + 3B_3^2B_1\cos^2(3\omega t + \varphi_3)\cos(\omega t + \varphi_1) + 3B_3^2B_5\cos^2(3\omega t + \varphi_3)\cos(5\omega t + \varphi_5) + \cdots \\ &\quad + 6B_1B_3B_5\cos(\omega t + \varphi_1)\cos(3\omega t + \varphi_3)\cos(5\omega t + \varphi_5) \\ &\quad + 6B_1B_3B_7\cos(\omega t + \varphi_1)\cos(3\omega t + \varphi_3)\cos(7\omega t + \varphi_7) + \cdots. \end{aligned} \quad \text{(A6)}$$

$$\begin{aligned} D_iE_i^2 &= (B_1\cos(\omega t + \varphi_1) + B_3\cos(3\omega t + \varphi_3) + \cdots) \\ &\quad \times (B_0 + B_2\cos(2\omega t + \varphi_2) + B_4\cos(4\omega t + \varphi_4) + \cdots)^2 \\ &= [B_1\cos(\omega t + \varphi_1) + B_3\cos(3\omega t + \varphi_3) + \cdots] \end{aligned}$$

$$\times [B_0^2 + B_2^2 \cos^2(2\omega t + \varphi_2) + B_4^2 \cos^2(4\omega t + \varphi_4) + \cdots$$
$$+ 2B_0B_2 \cos(2\omega t + \varphi_2) + 2B_0B_4 \cos(4\omega t + \varphi_4) + \cdots$$
$$+ 2B_2B_4 \cos(2\omega t + \varphi_2)\cos(4\omega t + \varphi_4) + 2B_2B_6 \cos(2\omega t + \varphi_2)\cos(6\omega t + \varphi_6)$$
$$+ \cdots]$$
$$= [B_1 \cos(\omega t + \varphi_1) + B_3 \cos(3\omega t + \varphi_3) + \cdots]$$
$$\times [(B_0^2 + \tfrac{1}{2}B_2^2 + \tfrac{1}{2}B_4^2 + \cdots) + \tfrac{1}{2}B_2 \cos(4\omega t + 2\varphi_2) + \tfrac{1}{2}B_4 \cos(8\omega t + 2\varphi_4) + \cdots$$
$$+ 2B_0B_2 \cos(2\omega t + \varphi_2) + 2B_0B_4 \cos(4\omega t + \varphi_4) + \cdots$$
$$+ B_2B_4 \cos(6\omega t + \varphi_2 + \varphi_4) + B_2B_4 \cos(2\omega t + \varphi_4 - \varphi_2)$$
$$+ B_2B_6 \cos(8\omega t + \varphi_2 + \varphi_6) + B_2B_6 \cos(4\omega t + \varphi_6 - \varphi_2) + \cdots]. \tag{A7}$$

Let $\cos[(2n_1 + 1)\omega t + \phi]$, $\cos[(2n_2 + 1)\omega t + \beta]$ and $\cos[(2n_3 + 1)\omega t + \gamma]$ (where n_1, n_2 and n_3 are 0, 1, 2, . . .) be three components of odd frequencies. Then, one has

$$\cos^3[(2n_1 + 1)\omega t + \phi] = \tfrac{1}{4}\cos[(6n_1 + 3)\omega t + 3\phi] + \tfrac{3}{4}\cos[(2n_1 + 1)\omega t + \phi]. \tag{A8}$$

$$\cos^2[(2n_1 + 1)\omega t + \phi]\cos[(2n_2 + 1)\omega t + \beta]$$
$$= \tfrac{1}{2}\cos[(2n_2 + 1)\omega t + \beta] + \tfrac{1}{4}\cos[(4n_1 - 2n_2 + 1)\omega t + 2\phi - \beta]$$
$$+ \tfrac{1}{4}\cos[(4n_1 + 2n_2 + 3)\omega t + 2\phi + \beta]. \tag{A9}$$

$$\cos[(2n_1 + 1)\omega t + \phi]\cos[(2n_2 + 1)\omega t + \beta]\cos[(2n_3 + 1)\omega t + \gamma]$$
$$= \tfrac{1}{4}\cos[(2n_1 + 2n_2 + 2n_3 + 3)\omega t + \phi + \beta + \gamma]$$
$$+ \tfrac{1}{4}\cos[(2n_1 + 2n_2 - 2n_3 + 1)\omega t + \phi + \beta - \gamma]$$
$$+ \tfrac{1}{4}\cos[(2n_1 - 2n_2 + 2n_3 + 3)\omega t + \phi - \beta + \gamma]$$
$$+ \tfrac{1}{4}\cos[(2n_3 - 2n_1 + 2n_2 + 1)\omega t - \phi + \beta + \gamma]. \tag{A10}$$

From the above equations, it is readily seen that D^3 belongs to the components of odd frequencies. Let $\cos(2m\omega t + \alpha_1)$ (where m is 0, 1, 2, . . .) represent one component of even frequencies. Then

$$\cos[(2n_1 + 1)\omega t + \phi]\cos^2(2m\omega t + \alpha_1)$$
$$= \tfrac{1}{2}\cos[(2n_1 + 1)\omega t + \phi] + \tfrac{1}{4}\cos[(2n_1 + 4m + 1)\omega t + \phi + 2\alpha_1]$$
$$+ \tfrac{1}{4}\cos[(2n_1 - 4m + 1)\omega t + \phi - 2\alpha_1]. \tag{A11}$$

Then, $D_iE_i^2$ also belongs to the components of odd frequencies. Similarly, it is also known that E_i^3 and $D_i^2E_i$ belong to the components of even frequencies. Thus, equation (A5) can be rewritten as

$$\ddot{D}_i + c\dot{D}_i + kD_i + \delta(D_i^3 + 3D_iE_i^2) = F\cos(\omega t + \alpha), \tag{A12}$$

$$\ddot{E}_i + c\dot{E}_i + kE_i + \delta(3D_i^2E_i + E_i^3) = 0. \tag{A13}$$

Clearly, if D_1 and E_1 satisfy equations (A12) and (A13), then D_l and $-E_l$ also satisfy the equations (A12) and (A13). Thus, the following theorem can be readily obtained:

Theorem: If the Duffing oscillator

$$\ddot{x} + c\dot{x} + kx + \delta x^3 = F\cos(\omega t + \alpha) \tag{A14}$$

is chaotic and $x_i(t) = D_i(t) + E_i(t)$ is one of the closed orbits, then $x_{\bar{i}} = D_i(t) - E_i(t)$ is another closed orbit, where

$$D_i(t) = B_1 \cos(\omega t + \varphi_1) + B_3 \cos(3\omega t + \varphi_3) + \cdots, \tag{A15}$$

$$E_i(t) = B_0 + B_2 \cos(2\omega t + \varphi_2) + B_4 \cos(4\omega t + \varphi_4) + \cdots. \tag{A16}$$

APPENDIX B: GROUP THEORY FOR SUB-HARMONICS

Now, consider the closed orbits with sub-harmonics. Let $x_s(t)$ be one closed orbit with sub-harmonics. $x_s(t)$ can be written as

$$x_s = B_0 + B_1 \cos(\Omega t + \varphi_1) + B_2 \cos(2\Omega t + \varphi_2) + B_3 \cos(3\Omega t + \varphi_3) + \cdots, \tag{B1}$$

where $\Omega = \omega/m$ and m is a positive integer. The case $m = 1$, will not be considered because it is the same case as in section 4.1. If $m > 1$, the sub-harmonics exist in $x_s(t)$. Similar to section 4.1, if the components of odd frequencies are defined as

$$D_s(t) = B_1 \cos(\Omega t + \varphi_1) + B_3 \cos(3\Omega t + \varphi_3) + B_5 \cos(5\Omega t + \varphi_5) + \cdots, \tag{B2}$$

and the components of even frequencies as

$$E_s(t) = B_0 + B_2 \cos(2\Omega t + \varphi_2) + B_4 \cos(4\Omega t + \varphi_4) + \cdots, \tag{B3}$$

one still can gets that D_s^3 and $D_s E_s^2$ belong to the components of odd frequencies; $D_s^2 E_s$ and E_s^3 belong to the components of even frequencies. Thus, one has

$$\ddot{D}_s + c\dot{D}_s + kD_s + \delta(D_s^3 + 3D_s E_s^2) = F \cos(\omega t); \tag{B4}$$

$$\ddot{E}_s + c\dot{E}_s + kE_s + \delta(3D_s^2 E_s + E_s^3) = 0. \tag{B5}$$

Consider some specific cases. If $m = 3, 5, 7, \ldots$, then D_s belongs to "the components of odd frequencies" and E_s belongs to "the components of even frequencies". Thus, it is readily seen from equations (B4) and (B5) that if $x_i(t) = D_s(t) + E_s(t)$ is one of the closed orbits, then $x_{\bar{s}}(t) = D_s(t) - E_s(t)$ is another closed orbit.

For $m = 2$,

$$D_s(t) = B_1 \cos(\tfrac{1}{2}\omega t + \varphi_1) + B_3 \cos(\tfrac{3}{2}\omega t + \varphi_3) + B_5 \cos(\tfrac{5}{2}\omega t + \varphi_5) + \cdots, \tag{B6}$$

$$E_s(t) = B_0 + B_2 \cos(\omega t + \varphi_2) + B_4 \cos(2\omega t + \varphi_4) + B_6 \cos(3\omega t + \varphi_6) + \cdots, \tag{B7}$$

Let $D'_s = E_s$ and $E'_s = D_s$. Similarly, if $x_i = D'_s + E'_s$ is one of the closed orbits, then $x_{\bar{s}} = D'_s - E'_s = E_s - D_s$ is another closed orbit.

For a closed orbit, not every point of the orbit can be an initial condition which can lead to this closed orbit. Without loss of generality, it is assumed that the initial condition is at $t = 0$. However, there are many initial conditions which can lead to the same closed orbit. An operator is defined as $N(x) = \ddot{x} + c\dot{x} + kx + \delta x^3$. Then, the Duffing oscillator is

$$N(x) = F \cos(\omega t). \tag{B8}$$

Suppose that

$$x_0 = \sum_{n=0}^{\infty} B_n \cos(n\Omega t + \varphi_n), \tag{B9}$$

is one solution for equation (B8), where $m\Omega = \omega$, then

$$N(x_0) = N\left(\sum_{n=0}^{\infty} B_n \cos(n\Omega t + \varphi_n)\right) = F \cos(\omega t). \tag{B10}$$

Furthermore,

$$N\left(\sum_{n=0}^{\infty} B_n \cos(n\Omega t + \varphi_n)\right) = F\cos(\omega t + 2\pi L), \tag{B11}$$

where $L = 0, \pm 1, \pm 2, \ldots$. Let $\omega t + 2\pi L = \omega t'$. Then $t = (\omega t' - 2\pi L)/\omega$. Thus, one has

$$N\left(\sum_{n=0}^{\infty} B_n \cos\left(n\Omega\left(\frac{\omega t' - 2\pi L}{\omega}\right) + \varphi_n\right)\right) = F\cos(\omega t'), \tag{B12}$$

or

$$N\left(\sum_{n=0}^{\infty} B_n \cos\left[n\Omega t' + \varphi_n - \frac{2\pi n L}{m}\right]\right) = F\cos(\omega t'). \tag{B13}$$

Therefore,

$$\sum_{n=0}^{\infty} B_n \cos\left[n\Omega t' + \varphi_n - \frac{2\pi n L}{m}\right]$$

is also a solution for equation (B8). L can be any integer. However, for distinctive solutions, one only needs to take $L = 0, 1, 2, \ldots, m$. Furthermore, if the components of even frequencies shift π phase, they are also the solutions. The π phase shift is written as

$$\sum_{n=0}^{\infty} B_n \cos\left[n\Omega t' + \varphi_n - \frac{2\pi n L}{m} + (m+n)\pi\right]. \tag{B14}$$

If m is odd, when n is even, the Fourier component belongs to the components of even frequencies, and $m + n$ is odd and $(m + n)\pi$ gives rise to π phase shift. When n is odd, the Fourier component belongs to the components of odd frequencies, and $m + n$ is even and $(m + n)\pi$ does not change the value of the Fourier component. Therefore, equation (B14) can be divided into D_s and E_s. Similarly, if m is even, (B14) can be divided into D_s' and E_s'. In either case, equation (B14) will satisfy (B4) and (B5). Thus, one has the following theorem.

Theorem: For Duffing oscillator

$$\ddot{x} + c\dot{x} + kx + \delta x^3 = F\cos(\omega t), \tag{B15}$$

if

$$x_0 = \sum_{n=0}^{\infty} B_n \cos(n\Omega t + \varphi_n) \tag{B16}$$

is one solution, then

$$x_L = \sum_{n=0}^{\infty} B_n \cos\left[n\Omega t' + \varphi_n - \frac{2\pi n L}{m} + (m+n)\pi\right] \tag{B17}$$

are also solutions, where $m\Omega = \omega$; m is a positive integer and $L = 0, 1, 2, \ldots, m - 1$.